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## BIOGRAPHY.

### BOLYAI JÁNOS. [JOHN BOLYAI.]

BY GEORGE BRUCE HALSTED.

FOR human thought, for culture, for intellectual freedom, for philosophy it is impossible to overestimate the tremendous importance of the non-Euclidean Geometry. The life of Bolyai János will therefore possess imperishable interest for mankind.

He was born December 15th, 1802, at Kolozsvár. His mother, Benkö Zsuzsánna, gifted, beautiful, imaginative, was devoted to her son. His father, Bolyai Farkas, whose biography is given in Vol. III., No. 1., of this MONTHLY, declared the son a marvel, a wonder-child.

Yet until long after both father and son had died, not one published word, except from the father, ever called the slightest attention to an achievement of genius by the son so far-reaching, so profound as to modify all present and future thought.

Yet the neglected János had the just appreciation given to genius of its own work. Speaking once of his own early youth he says: "The fore-knowledge was withheld from me that I should perfect the theory of mathematics and in this way bless the human race." In the great chest at Maros-Vásárhely wherein his papers are preserved I found an autobiographic fragment whick speaks of him as "the phœnix of Euclid." Yet no picture of him is known to exist, and I deemed it a discovery of interest to find from his passport that his eyes were blue.

From childhood great pains were taken with his education. The brigthest of his father's students taught him, except in mathematics, where the father himself was his teacher. His progress in mathematics was so lightning-quick, that, as his father often related, he waited not for the proof of the theorem or the solution of the problem, but gave them himself. "Like a demon sprang he on before me," says the father, "and bade me go further."

At twelve years of age he passed the "Rigorosum" covering six gymnasial classes, and according to the usage of that time became a "student." At this examination his translations from Magyar into Latin were written in the style of Tacitus, gaining the professor's unstinted praise.

János was two years a "student," or rather he for two years attended the college to play checkers. He was rarely present at the lectures. At the approach of the winter examinations, the professor of history, John Antal, later bishop of the reformed church, complained of him to his father. The father called up the idler and gave him some good advice, which he did not take. He went up to the examinetion after glancing through the lectures, and answered beautifully. Whatever and however much he was asked, he was always fully prepared.

Just so in the next semester he carried on his checker-playing ; but now the professor said nothing, and again the examination was a brilliant success.

When the father was sick he sent his son to teach the college classes in his stead ; and the students preferred the teaching of the thirteen-year-old János to that of the father—they got more out of it. Even in his twelfth year he played the violin so well, that he was able to execute the most difficult pieces at sight.

At that time was given perhaps the first opera in Maros-Vásárhely. At the presentation the blind composer was there. A young Saxon played first violin, János second. During the representation a string broke on the violin of the prima, the Saxon. They exchanged the notes, and the child János played at sight first violin. The blind composer, who had hitherto been dissatisfied, cried out, "Bravo ! now the prima dominates !"

At the age of fifteen János went away to the great Engineering Academy in Austria. Not long after, the Archduke John visited the Academy as inspector, and asked that some of the new students be called upon. When the choice fell on young Bolyai, he quickly worked out the problem given by the Professor, and went on to the next, and so on. The Archduke was astonished at the genius of the youth, interrupted him with praises, and said to the Professor, "The other students should be put under this one ; he knows more than the whole class."

After this, Magnates came from Erdély came to Vienna. At an audience with the Archduke he told them what a perfect genius of a boy from Erdély was at the Academy. He asked if they knew his father. They answered, the father was also a genius and their friend. The Archduke sent him word that he was delighted with his son, who if he would behave himself might count on a great career. János finished the five years course with the highest distinction, and at twenty entered as Cadet the Engineering Corps. At twenty-one he was Lieuten-

ant. In the whole army he was the first as mathematician, first as violinist, but also first with his sabre. He fought duels continually, several with fatal ending, he always victor. Once he was challenged by thirteen officers of a cavalry regiment. He accepted all with the condition that he was to be allowed to play a piece on his violin after every two duels. As always he was victor in all.

His conduct was haughty. But remember, in less than three months from his graduation he had, as he says in his letter of November 3d, 1823, "*from nothing created another wholly new world.*" He had solved the problem of the ages, the problem of parallels, and had made a new kind of universal space.

In August, 1823, he was graduated at the royal Engineering Academy as Underlieutenant, and ordered for service to Temesvár. He left Vienna on September 17th and arrived at Temesvár on September 30th. His extraordinary creation falls thus in the very first month of his service, and before his twenty-first birthday.

The splendid youth, who knew that he had succeeded where the very greatest of his predecessors on earth had failed, could scarcely have been expected to submit to any affront. His sabre was ready for whoever wished to impale himself.

Franz Schmidt has preserved a glimpse of János at Temesvár in the story often told him by his father, Anton Schmidt, of a young officers of engineers with whom he feared to come in contact, who to prove the might of his arm and the temper of his Damascus blade, was accustomed to show his visitors how with one blow he could cut off a heavy nail driven into his door-post. But not even this trenchant blade could open any brain about him to appreciation of his immortal achievement.

Only his father knew and publicly declared what he had done.

János has left on record that it was in the Engineering Academy that he began to investigate the properties of the line everywhere equidistant from a straight line, on the hypothesis that it was not straight, and on that investigation corresponded with his father in 1820. But not until 1823 did he completely penetrate and master the whole subject. He wrote out a treatise on the subject, which in 1825 he transmitted to his former professor Johann Wolter von Eckwehr.

His father Bolyai Farkas induced him to translate this into Latin and issue it as an Appendix to the first volume of the father's "Tentamen." It was printed separately and appeared before the volume with which it was afterward bound up.

After years, 1843, in the second edition of his Arithmetic (in Magyar) the father says : "The *Appendix* is a work worth whole folios. It is to the true geometer a work beautiful, necessary, original, colossal. How many thinkers have even up to the latest time endeavored in vain to make sure the one foundation of the edifice of Euclidean geometry ? In the mentioned work is built up an absolute and for all cases true geometry. Few have appreciated its worth, yet it is in one word a true classic."

After more than half a century the world has come to that opinion. Says F. S. Macauley in 1897, "I was simply fascinated with your translation of Bolyai's 'Science Absolute' when I first read it, as indeed almost any one would be who comes across it."

## A NEW SOLUTION OF THE CUBIC EQUATION.

By DR. L. E. DICKSON.

The method here suggested is analogous to that used by Ferrari in solving the biquadratic. I multiply the reduced cubic by $x-c$ and determine $c$ so that the resolvent cubic shall be binomial. Thus, adding $(ax+b)^2$ to each member of

$$(x-c)(x^3+px+q)=0$$

and requiring the new equation to take the form

$$[x^2-(c/2)x+\lambda+(p/6)]^2=(ax+b)^2 \tag{1}$$

we have the conditions on $a$, $b$, $\lambda$ :

$$\begin{aligned} a^2&=2\lambda+\tfrac{1}{4}c^2-\tfrac{2}{3}p \\ 2ab&=-c\lambda+\tfrac{5}{6}cp-q \\ b^2&=(\lambda+\tfrac{1}{12}p)^2+qc. \end{aligned} \tag{2}$$

Since $c$ is a root of (1), we may set

$$\tfrac{1}{2}c^2+\lambda+\tfrac{1}{6}p=ac+b, \tag{3}$$

limiting ourselves to one of the two sets of values for $a$ and $b$ in terms of $\lambda$ and $c$. Equating the two values for $4a^2b^2$ given by (2), we reach the resolvent cubic for $\lambda$ in *reduced* form :

$$8\lambda^3+2\lambda(3cq+c^2p-\tfrac{1}{3}p^2)+(qc^3-cpq-q^2-\tfrac{2}{3}p^2c^2-\tfrac{2}{27}p^3)=0. \tag{4}$$

Its discriminant is found to be

$$R(c^3+pc+q)^2, \text{ where } R\equiv\tfrac{1}{4}q^2+\tfrac{1}{27}p^3$$

is the discriminant of the given cubic.

The coefficient of $2\lambda$ in (4) vanishes for

$$\tfrac{1}{3}pc=-\tfrac{1}{2}q+\sqrt{R}, \tag{5}$$

whence

$$c^3+pc+q=\frac{108\,R}{p^3}(-\tfrac{1}{2}q+\sqrt{R}).$$

Hence, if we determine $c$ by (5) the cubic (4) becomes

$$8\lambda^3=-q(c^3+pc+q)+8R=\left(\frac{6\sqrt{R}}{p}\right)^3(-\tfrac{1}{2}q+\sqrt{R}). \tag{6}$$

Then by (2), $a^2=\frac{6\sqrt{P}}{p}(-\frac{1}{2}q+\sqrt{R})+\frac{1}{4}c^2-\frac{2}{3}I$.

By a slight calculation, we find

$$a=\tfrac{1}{2}c+\sqrt[3]{-\tfrac{1}{2}q+\sqrt{R}}+\sqrt[3]{-\tfrac{1}{2}q-\sqrt{R}}.$$

This would be expected, since by (3)

$$(x-c)(x-a+\tfrac{1}{2}c)=x^2-\tfrac{1}{2}cx+\lambda+\tfrac{1}{6}p-ax-b,$$

whence $a-\frac{1}{2}c$ is a root of the given cubic.

*University of California, January 17.*

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## THE FIRST AWARD OF THE LOBACHEVSKI PRIZE.

By GEORGE BRUCE HALSTED, A. M. (Princeton); Ph. D. (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

The Lobachévski prize is adjudged every three years. Its value is five hundred roubles. It is given for work in geometry, preferably non-Euclidean geometry. All works published within the six years preceding the award of the prize and sent by their authors to the Physico-Mathematical Society of Kazan are allowed to compete if published in Russian, French, German, English, Italian, or Latin.

The Society has now in formal session awarded the prize to Sophus Lie, Professor of Mathematics at the University of Leipzig, for his work "Theorie der Transformationsgruppen. Band III. Leipzig, 1893." In this work the theory of non-Euclidean geometry has been exhaustively re-stated and re-established in a profound investigation of the work of Helmholtz on the space-problem.

To the genius of Helmholtz is due the conception of studying the essential characteristics of a space by a consideration of the movements possible therein.

But since the time when Helmholtz did his work on this subject, the greatest of living mathematicians, Sophus Lie, formerly of Christiana, has enriched mathematics with a new instrument, the Theory of Groups, which its creator has applied with tremendous power to the Hemholtz treatment. Lie finds, as was almost inevitable, that certain details had escaped the great physicist, but that, with the tact of true genius, he had kept his main results free from error, though there comes to light a superfluity in his explicit assumptions, an unconscious assumption now seen to be mathematically important for the rigor of the demonstration, and at least one definits error in minor results.

Lie's method is in general the following. Consider a tridimensional space, in which a point is defined by three quantities, $x$, $y$, $z$.

A movement is defined by three equations :

$$x' = f(x, y, z) \; ; \quad y' = \varphi(x, y, z) \; ; \quad z' = \psi(x, y, z).$$

By this transformation an assemblage $A$ of points $(x, y, z)$ becomes an assemblage $A'$ of points $(x', y', z')$.

This represents a movement which changes $A$ to $A'$. Now make, in regard to the space to be studied, the following assumptions :

1st. Assume : in reference to any pair of points which are moved, there is *something* which is left unchanged by the motion. That is, after an assemblage of points $A$ has been turned by a single motion into an assemblage of points $A'$, there is a certain function $F$ of the coördinates of any pair of the old points $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$ which equals that same function $F$ of the corresponding new coördinates $(x_1', y_1', z_1')$, $(x_2', y_2', z_2')$ ; that is $F(x_1, y_1, z_1, x_2, y_2, z_2) = F(x_1', y_1', z_1', x_2', y_2', z_2')$. This *something* corresponds to the Cayley definition of the distance of two points when interpreted as completely independent of ordinary measurement by superposition of an unchanging sect as unit for length.

This independence, involving the determination of cross-ratio without any use of ordinary ratio, without using congruence, without using motion, Cayley never clearly saw. It follows from the profound pure projective geometry of von Staudt.

2nd. Assume : if one point of an assemblage is fixed, every other point of this assemblage, *without any exception*, describes a surface (a two-dimensional aggregate).

When two points are fixed a point in general (exceptions being possible) describes a curve (a one-dimensional aggregate). Finally, if three arbitrary points are fixed, all are fixed (exceptions being possible). With these assumptions Lie proves exhaustively that the general results of Helmholtz and Riemann follow : that is, there are three and only three spaces which fulfill these requirements, namely, the traditional or Euclidean space, and the spaces in which the group of movements possible is the projective group transforming into itself one or the other of the surfaces of the second degree $x^2 + y^2 + z^2 \pm 1 = 0$.

In the appreciation of this work of Lie's prepared for the Society by Felix Klein, for which the Lobachévski gold medal was given him, he says that Lie's work stands out so prominently over all the others to be compared with it that a doubt as to the award of the prize would scarcely have been possible. Decisive for this judgment as to the height of the scientific achievement is not only the extraordinary depth and keenness with which Lie in the fifth section of his book handles what he has called the Riemann-Helmholtz space problem, but especially the circumstance that this treatment appears so to say as logical consequence of Lie's long-continued creative work in the province of geometry, especially his theory of continuous transformation-groups.

The extraordinary importance which the works of Lie possess for the general development of geometry can scarcely be over-estimated. In the coming years they will be still more widely prized than hitherto. Passing then to the consideration of the present state of the space question, Klein takes up the origin

of axioms. Whence come the axioms ? A mathematician who knows the non-Euclidean theories would scarcely maintain the position of earlier times that the axioms as to their concrete content are necessities of the inner intuition.

What to the uninitiated appears as such necessity, shows itself after long occupation with the non-Euclidean problems as result of very complex processes, and especially education and habit.

Do the axioms come from experience ? Helmholtz energetically says yes ! as is well known. But his expositions seem in a definite direction incomplete. One will, in thinking over these, willingly admit that experience plays an important part in the formation of axioms, but will notice that just the point especially interesting to the mathematician remains untouched by Helmholtz.

It is the question of a process which we always complete in exactly the same way in the theoretical handling of any empirical data, and which therefore may seem quite clear to the scientist.

Expressed generally : *always the results of any observations hold good only within definite limits of precision and under particular conditions ; when we set up the axioms, we put in the place of these results statements of absolute precision and generality.*

In this "idealizing" of empirical data lies in my opinion the peculiar essence of axioms. Therein our addition is limited in its arbitrariness at first only by this, that it must cling to the results of experience and on the other hand introduce no logical contradiction.

Then enters as regulator also that which Mach calls the "economy of thinking." No one will rationally hold fast to a more complicated system of axioms when he sees, that with a simpler system he already completely attains the exactitude requisite to the representation of the empirical data.

Klein goes on to mention the possibility of a series of topologically distinguishable space-forms built of limited (simply compendent) space-pieces either all Euclidean, all Lobachévskian, or all Riemannian. Beside these three just mentioned family-types, the parabolic, the hyperbolic, the single elliptic, Killing has shown that the spherical, in which two geodetics always cut in two points, is the only one which as a whole is freely movable in itself.

Then Klein says : I consider all the topologically distinguished space-forms as equally compatible with experience. That in our theoretic considerations we prefer some of these space-forms (namely the family types, that is the properly parabolic, hyperbolic, elliptic) in order to finally assume the parabolic geometry, that is the customary Euclidean geometry as valid, happens simply from the principle of economy.

# DEPARTMENTS.

## SOLUTIONS OF PROBLEMS.

### ALGEBRA.

78. Proposed by J. MARCUS BOORMAN, Consultative Mechanician and Counselor at Law, Woodmere, Long Island, N. Y.

Solve $x^2+xy=10$.....(1); $y^2+xy=15$.....(2), for all the roots, and prove that they are the roots.

I. Summary of Solutions by J. OWEN MAHONEY, M. Sc., Lynnville, Tenn.; F. R. HONEY, Ph. B., Instructor at Trinity College, New Haven, Conn.; J. SCHEFFER, A. M., Hagerstown, Md.; G. B. M. ZERR, A. M., Ph. D., The Russell College, Lebanon, Va.; P. S. BERG, A. M., Principal of Schools, Larimore, N. D.; A. H. BELL, Hillsboro, Ill.; H. C. WILKES, Skull Run, W. Va.; CHARLES C. CROSS, Laytonsville, Md.; I. H. BRYANT, A. M., Ft. Smith High School, Ft. Smith, Ark.; and COOPER D. SCHMITT, A. M., University of Tennessee, Knoxville, Tenn.

Divide (1) by (2), then $x/y=\frac{2}{3}$..... (3); $x$ and $y$ must be plus or minus together. Then $x$ from (3) in (1), gives $y^2=9$, or $y=\pm 3$. Also, $x(=2y/3)=\pm 2$. $\therefore$ $x=\pm 2$, $y=\pm 3$ are the roots, which can be proved by direct substitution. [MAHONEY, HONEY, SCHEFFER, AND BELL.]

Adding (1) and (2), and extracting square root, $x+y=\pm 5$.....(4). Then (3) in (4) gives, $y=\pm 3$, $x=\pm 2$. [ZERR.]

Subtracting (2) from (1), $x^2-y^2=-5$.....(5). Then (5) by (4), $x-y=\mp 1$.....(6). From (4) and (6), $x=\pm 2$, $y=\pm 3$. [BERG, AND WILKES.]

Solve (1) for $x$, then $x=[-y\pm\sqrt{(40+y^2)}]/2$ ....(7). Substituting (7) in (2), and reducing, $y=\pm 3$. Similarly, $x=\pm 2$. [CROSS.]

Substitute $vy$ for $x$ and solve; then $v=\frac{3}{2}$ or $-1$. By substituting and reducing, $x=\pm 2$, and $y=\pm 3$. Then if $v=\frac{3}{2}$, $x=\pm 2$, $y=\pm 3$. If $v=-1$, $x^2=\infty$, $x=\pm\infty$, $y=\mp\infty$. The first values satisfy the equations. Substitute the second values, $x^2=\infty$, $x=+\infty$, $y=-\infty$, $xy=-\infty$, $x^2+xy=\infty-\infty$. Since $\infty-\infty$ is an indeterminate expression, it may equal any numbers. Therefore the equations are satisfied for the last values of $x$ and $y$ as well as for the first values. [BRYANT.]

From (1), $y=(10-x^2)/x$. In (2), we get $10-x^2+(100-20x^2+x^4)/x^2=15$. Whence $25x^2-100=0$.....(8); or $x=\pm 2$. Being an equation of the fourth degree we ought to have four answers. We can write (8), $0x^4+0x^3+25x^2-100=0$. Since coefficients of two highest powers are zero, this indicates two infinity roots, which may be *claimed* defective in solutions in print. [SCHMITT.]

See *Analyst*, Vol. VIII, page 111, and Vol. IX, page 53, as well as solution below, for discussion as to whether $x=\pm\infty$ and $y=\mp\infty$ are also roots of the equations. [EDITOR.]

II. Solution by the PROPOSER.

The equations are fourth degree in $x$; $y$; the *singular* case in $x^2+xy=a$ .....$(A)$; $y^2+exy=A+d$..... $(B)$. *sub-ultimate* fourth degree when the positive are the negative roots reversed in signs. Here $a=10$; $A=15$; $e$ (*variable*) $=1$; $d=F(e-1)=0$.

Transpose (I), $x^2=10-xy$; then transpose (II), and multiply, giving $x^2y^2=150-25xy+x^2y^2$.....(III). $\therefore xy=6$.....(IV), and $(1-1)x^2y^2-25xy+(12\frac{1}{2})^2[1/(1-1)]=(12\frac{1}{2})^2/(1-1)-150$.....(III$a$). By (IV) and (I), (II); $x^2=4$; $y^2=9$; $\therefore x=\pm2$; to $y=\pm3$; four *true* roots.....(V). But $x^2y^2$ (vanished) is quadratic; it has therefore a *second* root $-xy=-6$; hence in (I), $x^2-(-6)=10$; *i. e.* $x^2=10+(-6)$.....(VI). $\therefore$ extract negatively $x=\mp2$; $y=\mp3$ [by parity in (II)]; and $x=\pm2$; $y=\pm3$; .....(V) are the eight required roots of Case (I), (II) and are *all* its roots. Q. V. D.

Presumably (*visum*) demonstrated.

Proof. Case (I) (II) is *biquadrate* because $xy=6$ that yields *two* pairs of roots flows *equally from* both only possible extractions of the *quadrate* $x^2y^2$..... $\therefore -xy=6$ *must* yield, failing other derivatives of (III), also the second set of roots. For now obviously (VI) cannot be $x^2=16$ to $y^2=21$ (nor $y^2=9$); because then (I) a whole number $16+4\sqrt{(21)}=10$; or an irrational surd=a rational number, *i. e.* reducing $\sqrt{(21)}=\mp\frac{3}{2}$, which is *not* true. Nor can $x=\int\sqrt{(-1)}$; if so therefore $y=\int[-\sqrt{(-1)}]$; to make $xy$ positive). $\therefore xy-\int^2 I=10$.... is (I); $xy-\int^2 I=15$ is (II) which *cannot* be for 10; 15; both positive. If it can be then ..... (1); (II) become: (I) $xy-x^2=10$ and (II) $xy-y^2=15$; then must $\mp y$; to $\pm x$; be true. Deduct (I) from (II), $\therefore x^2-y^2=5=st$....(VIII). Let $x+y=s$; $\therefore x-y=t$; add, $\therefore x=\frac{1}{2}(s+t)$; $y=\frac{1}{2}(s-t)$; $\therefore xy=\frac{1}{4}(s^2-t^2)$. Multiply by 4, change signs, etc., $\therefore x^2=\frac{1}{4}(s^2+2st+t^2)$ put into (I), $s^2+2st+t^2-4xy=-40$ ....(IX), [$\pm$ are changed by $\sqrt{(-1)}$]; and $y^2=\frac{1}{4}(s^2-2st-t^2)$ put into (II), $s^2-2st+t^2-4xy=-60$. Add the 2 first, $\therefore x^2+y^2=\frac{1}{2}(s^2+t^2)$, replace in (I); then $2x^2-4xy+2y^2+2st=-40$; but (VIII), $st=5$; $\therefore (x-y)^2=-20-5=-25$; *i. e.* $x-y=\pm5\sqrt{(-1)}$.....(X), and by (VIII) $x+y=\mp\sqrt{(-1)}$ .... (XI); add (X), (XI), $\therefore 2x=\pm4\sqrt{(-1)}$; $x=\pm2\sqrt{(-1)}$; and by (XI)$-$(X) $y=\mp3\sqrt{(-1)}$; *which do not* satisfy (I), (II). $\therefore x$; $y$ are not imaginary roots of (I) (II) and equation (IX) is falsely put as equation (I). But correct the false factors $\sqrt{(-1)}$ in (IX), (X), (XI) and they by same process yield the roots of (I) (II) found.

Second (see note).

(I) (II) have *no sort* of unreal roots. If they may $x=a+ei$; $y=b+ci$; $\therefore x^2=a^2-e^2+2aei$ and $y^2=b^2-c^2+2bci$. $\therefore ab-ec+(ac+be)i=xy$....($N$), and $b^2-a^2+e^2-c^2=5=$(II)$-$(I).....($P$); because to cancel $i$ (I) (II) we *must have*, $2aei=(ac+be)i=2bci$.....($Q$). For *else* rationals 10; 15; $(a^2-e^2)$, etc., *have to* equal $f\sqrt{(-1)}$ or $f\sqrt{(-1)}$; etc., *i. e.* real numbers *can be* partly *un*-real, which is absurd! $\therefore$ by ($Q$) $2a=b+ac/e$; and $2b=a+be/c$..... [These easily reduce to $c=\int i$; $e=\int i$..... multiplied by $i$. $\therefore ci=fii$; a *real* number. So our postulate that *ci can be un*-real is false.] $\therefore 2b-a=be/c$ and $2a-b=ac/e$. $\therefore b(2-e^2/ec)=a$ and $a(2-c^2/ec)=b$. [Thence $b<a$ and $b>a$! if $c=e$, as here

proven.] Multiply $\therefore ab[4-2(c^2+e^2)/ec+1]=ab$ ; cancel $ab=ab$, reduce, etc., $\therefore 2ce=c^2+e^2$, $\therefore (c-e)^2=0$.....$(K)$. $\therefore c=e$; ..... $\therefore (P_{|})$ is $b^2-a^2=5$ and our assumed *un*-real $ci$ ; $ei$ ; *do not exist*. $\therefore x=a_{|}$ ; $y=b_{|}$ *real* numbers in $(N)$ $(Q)$ just as we found above.

Again, (III$a$) reduced is $\sqrt{_{|}}(1-1)xy=[\pm 1/\sqrt{_{|}}(1-1)]\{[12\frac{1}{2}\pm\sqrt{_{|}}[156\frac{1}{4}-150(1-1)]\}$.....(XII). $\therefore xy=[\pm 1/(1-1)](12\frac{1}{2}\pm 2\frac{1}{2})$. $\therefore x_2y_2=\pm 15/(1-1)$ ; $x_ay_a=\pm 10/(1-1)$.....(XIII). [Unless in $\int 1/(1-1)$ of (XII); $-150(1-1)=0$, for which value $xy=[1/(1-1)](12\frac{1}{2}\pm 12\frac{1}{2})=x_3y_3=25/(1-1)$, or $x_4y_4=12\frac{1}{2}[(1-1)/(1-1)$.....(XIV)]. Hence by (XII) and (I), (II) $x_a^2=\pm 10[1-1/(1-1)]$ ; $y_a=\pm 10[\frac{3}{2}-1/(1-1)]$ ; $x_2^2=\pm 10[1-1.5/(1-1)]$ ; $y_2^2=\pm 15[1-1/(1-1)]$ ; multiply $x_2$ ; $y_2$ ; $\therefore x_2y_2=\sqrt{_{|}}\{15[10-10/(1-1)]\}$ ; (or ?) .....(XV) ; but (XV) *is not* equation (XIII) nor does it nor (XIV) any way satisfy (I) (II). $\therefore x_a, y_a, x_2, y_2$, are *not*, either as by (XIII) or (XIV) roots of (I) (II). Q. E. D.

Last, put $r=$ ratio $y:x$. $\therefore y=rx$ ; $x^2(1+r)=10$.....(I$a$) ; $x^2(r^2+r)=15$ .... .($H$). $\therefore r=\frac{3}{2}$ ; $r_{|}=-1$.....($K$). Ratio $\frac{3}{2}$ gives $x=\pm 2$ to $y=\pm 3$..... (V), or $x=\mp 2$ ; $y=\mp 3$.....(VI) above. But by $r_{|}=-1$ ; (I$a$) gives $x^2(1-1)=10$ ; $x_5=\pm\sqrt{_{|}}[10/(1-1)]$; and ($H$) $x_0=\pm\sqrt{_{|}}[15/(1-1)]$ ; $\therefore y_5=\pm\sqrt{_{|}}[10/(1-1)]$; $y_0=\mp\sqrt{_{|}}[15/(1-1)]$ ; .....($M$). Whether or no ($M$) be the quasi roots of (XIV) *none* satisfy (I) (II)! Besides they are *too many* and give (I)(II) 10, 14 or 18 roots! So quasi-results (XV) ($M$) *are not* roots and ratio $r_{|}=-1$ yields *no* root. $\therefore$ ratio $\frac{3}{2}$ covers all eight roots of (I) (II), viz. roots (V) direct, or (V) with contrary signs as above. Q. V. D.

Finally, solve (I) (II) *generally*. $\therefore x=\pm a/\sqrt{_{|}}(a+A)$ ; $y=\pm A/\sqrt{_{|}}(a+A)$ ; $xy=aA/(A+a)$. Now $d=(e-1)[(aA)/(a+A)]$, so equation ($A$) is equation (I), but ($B$) is $y^2+exy=A+[aA/(a+A)](e-1)$.....($L$); hence $x^2y^2=Aa+[a^2A(e-1)/(a+A)]-(ae+A)xy+ex^2y^2$ ... is (III$a$) *by generalizing*, and therefore $x^2y^2-[a+(A+1)/(e-1)]xy+a^2A/(a+A)+aA/(e-1)=0$ is the general form .....($J$). $\therefore$ take ($A$) $x^2=a-xy$ and ($B$) $y^2=A+d-exy$; and let $e=2$. $\therefore d=(e-1)xy=xy=6$ ; $A=15$ ; $y^2+2xy=21$.....($B_{|}$) ; $(2-1)x^2y^2-41xy+210=0$.....($C$). $\therefore (xy-(20\frac{1}{2})^2=210\frac{1}{4}$.....($C_{|}$). $\therefore xy=20\frac{1}{2}\mp 14\frac{1}{2}$, *i. e.* $xy=6$ .....(IV), same as by (I) (II).

But also $x_{||}y_{||}=20\frac{1}{2}+14\frac{1}{2}=35$.....($D$). $\therefore$ by ($A$) or (I) $x_{||}^2=10-35=-25$ ; *while* by ($B_{|}$) $y^2=21-70=-49$. $y_{||}=\mp_{|}7\sqrt{(-1)}$ ; $x=\pm_{|}5\sqrt{(-1)}$.... ($E$) roots that *do* satisfy (I) or ($A$) ; ($B_{|}$) *as well as* $x=\pm_{|}2$ ; $y=\pm_{|}3$.....(V), —and a general sub-ultimate fourth degree ($A$) ($B_{|}$) has, its eight roots in pairs "where the positive are the negative roots *reversed in signs*."

Half are roots *both of* ($A$) ($B_{|}$) and of (I) (II) because (I) is *identically* equation ($A$). But changing ($B_{|}$) in (II) changes its *second sets* of roots,—to say it *destroys* them is absurd! So therefore (I) (II) have *their own half set* of co-roots besides the roots that (II) shares also with ($B_{|}$) because (I) is *identically* ($A$). Q. E. D.

As the second set *cannot be unreal*, they *are real* and derive of $x^2y^2$ *vanished* by *symmetry* which is *why* (III$a$) *cannot* diverge its roots (VI) from (V) as

equation ($C$) *does* diverge its roots ($E$) from the roots (V). Therefore, *no other result being possible*, the roots that (II) has with (I) but not with ($B_1$) are $x_1 = \mp_1 2$; $y_1 = \mp_1 3$ and these with $x = \pm_1 2$; $y = \pm_1 3$ are *the eight*, and are *all* the roots of (I) (II) a fourth degree *singular* case. Q. E. D.

NOTE.—The $\pm_1 \ldots \pm_1$; $\pm_1 \ldots \mp_1$; or $(_1)\ldots(_1)$; mean "change signs in *unison only*." Thus in "SECOND" $_1a \ldots _1b$ (*not a* prime, etc.)

## GEOMETRY.

**81. Proposed by CHAS. C. CROSS, Laytonsville, Md.**

A circle is drawn bisecting the lines joining the points of contact of the escribed circles with the sides produced. Another circle is drawn passing through the centers of the circles drawn tangent externally to the in-circle and internally to the sides of the triangle. Prove that the centers of these two circles, the in-center and the circumcircle are collinear.

**I. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.**

Let $ABC$ be the triangle; $O_a$, $O_b$, and $O_c$ the centers of the escribed circles tangent externally to the sides $a$, $b$, and $c$ respectively, $h$ and $g$ the points of tangency of circle whose center is $O_a$ with the sides $c$ and $b$ produced; $e$ and $f$ the points of tangency of the circle, center $O_b$, with the sides $c$ and $a$ produced; $d$ and $k$ the points of tangency of the circle, center $O_c$, with the sides $b$ and $a$; $O$ the center of the in-circle; $a$, $b$, and $c$ the centers of the circles described tangent to circle, center $O$, and the sides $b$ and $c$, $c$ and $a$, and $a$ and $b$; $E$, $G$, and $K$ the feet of the perpendiculars from the centers $a$, $b$, and $c$ to the side $a$; $P$, $Q$, and $R$ the middle points of the lines $de$, $gh$, and $hk$ respectively; $F$, $L$, and $D$ the feet of the perpendiculars let fall from $P$, $Q$, and $R$ on the side $a$; and $O'$, $M'$, $M$ the centers of the circles through $a$, $b$, and $c$, $A$, $B$, and $C$, and $P$, $Q$, and $R$ respectively.

Lemma—The coördinates of the center of a circle passing through $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$, are given by

$$\alpha = \frac{(x_1^2+y_1^2)(y_3-y_2)+(x_2^2+y_2^2)(y_1-y_3)+(x_3^2+y_3^2)(y_2-y_1)}{2x_1(y_3-y_2)+2x_2(y_1-y_3)+2x_3(y_2-y_1)} \quad \ldots\ldots\ldots\ldots (1).$$

$$\beta = \frac{(x_1^2+y_1^2)(x_3-x_2)+(x_2^2+y_2^2)(x_1-x_3)+(x_3^2+y_3^2)(x_2-x_1)}{2y_1(x_3-x_2)+2y_2(x_1-x_3)+2y_3(x_2-x_1)} \quad \ldots\ldots\ldots\ldots (2).$$

$Bk = Cf = s-a$, $Cg = Ad = s-b$, $Bh = Ae = s-c$.

Taking $B$ as origin and axes rectangular we get coördinates of $M$, $\{\frac{1}{2}a, \frac{1}{2}a\cot A\}$; of $O$, $\{s-b, r\}$; of $k$, $\{-[s-a], 0\}$; of $h$, $\{-[s-c]\cos B, -[s-c]\sin B\}$; of $f$, $\{s, 0\}$; of $g$, $\{a+[s-b]\cos C, -[s-b]\sin C\}$; of $d$, $\{s\cos C-a, s\sin C\}$; of $e$, $\{s\cos B, s\sin B\}$.

$\therefore$ of $R$, $\{-[\frac{1}{2}(s-a)+\frac{1}{2}(s-c)\cos B],\ -\frac{1}{2}(s-c)\sin B\}$;

of $Q$, $\{\frac{1}{2}[s+a+(s-b)\cos C],\ -\frac{1}{2}(s-b)\sin C\}$;

of $P$, $\{\frac{1}{2}[s(\cos C+\cos B)-a],\ \frac{1}{2}s(\sin C+\sin B\}$.

$BH=r\cot\frac{1}{2}B$, $OH=r$, $bE=x$, $BE=x\cot\frac{1}{2}B$.

$\therefore$ $(r-x)^2\cot^2\frac{1}{2}B+(r-x)^2=(r+x)^2$. $\therefore$ $x=r(1-\sin\frac{1}{2}B)/(1+\sin\frac{1}{2}B)$.

Let $r(1-\sin\frac{1}{2}B)/(1+\sin\frac{1}{2}B)=m$, $r(1-\sin\frac{1}{2}C)/(1+\sin\frac{1}{2}C)=n$, $r(1-\sin\frac{1}{2}A)/(1+\sin\frac{1}{2}A)=l$.

$\therefore$ coördinates of $b$ are $(m\cot\frac{1}{2}B,\ m)$; of $c$, $(a-n\cot\frac{1}{2}C,\ n)$; of $a$, $(p\cos aBC,\ p\sin aBC)$, where $p=\sqrt{[c^2-2cl\cot\frac{1}{2}B+l^2\operatorname{cosec}^2\frac{1}{2}A]}$, and $\tan aAB=(c\tan B-l\tan B\cot\frac{1}{2}A-l)/(c-l\cot\frac{1}{2}A+l\tan B)$.

Substituting in (1) and (2) the truth of the proposition appears.

By substituting the coördinates of $P$, $Q$, $R$, and $a$, $b$, $c$ in (1), (2) we get, after a prodigious amount of work, the coördinates of two points. If the line through these two points coincides with the line through $O$, $M$, the proposition is true.

[NOTE.—Dr. Zerr furnished a very beautiful figure to go with his solution, but we lacked the time to engrave it. EDITOR.]

**83. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in Ohio University, Athens, O.**

$\theta$ being variable, find the locus of a point whose coördinates are

$$a\tan(\theta+\alpha),\quad b\tan(\theta+\beta).$$

**Solution by the PROPOSER.**

The rectilinear coördinates being $x$ and $y$, $x=a\tan(\theta+\alpha)$..... .....(1),

$y=b\tan(\theta+\beta)$.....(2). (1) gives $\theta+\alpha=\tan^{-1}(x/a)$.....(3),

$\theta+\beta=\tan^{-1}(y/b)$....(4). Eliminating $\theta$, $\tan^{-1}(x/a)-\tan^{-1}(y/b)=\alpha-\beta$....(5).

Taking tangents of both members of (5) and reducing,

$$xy-\cot(\alpha-\beta)(bx-ay)+ab=0.....\quad (6),$$

the equation to the required locus.

Solved in a similar manner by *COOPER D. SCHMITT, T. W. PALMER, OTTO CLAYTON*, and *G. B. M. ZERR*.

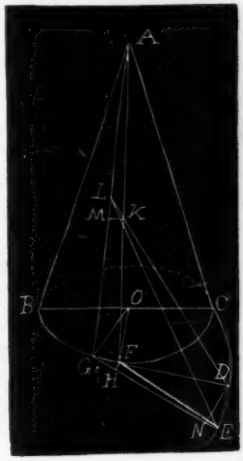
## CALCULUS.

**65. Proposed by GEORGE LILLEY, Ph. D., LL. D., Professor of Mathematics, State University, Eugene, Ore.**

A string is wound spirally 100 times around a cone 100 feet high and 2 feet in diameter at the base. Through what distance will a duck swim in unwinding the string keeping it taut at all times, the cone standing on its base and at right angles to the surface of the water?

**Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.**

Let $h$=height, $r$=radius of base, $l=\sqrt{(h^2+r^2)}$=slant height, $n$=number of spirals; $D$, $E$, two consecutive points in the duck's path; $K$, $L$, corresponding consecutive points of tangency of string to cone; $F$, $G$, corresponding consecutive points in base of cone.




Let $CD=s$, $FK=x$. $\therefore$ $DE=ds$, $ML=dx$.

$ds^2=DN^2+NE^2$ .....(1).

From the similar triangles $GOF$ and $DFE$,

$r : FG=FD : DN$. $\therefore$ $DN=(FD.FG)/r$.....(2).

From the triangles $LMK$ and $KFD$,

$dx : MK=x : DF$. $\therefore$ $FD=(x.MK)/dx$..... ....(3).

$MK : (l-x)=GF : l$. $\therefore$ $MK=[(l-x).GF]/l$.....(4).

But $CF : x=2\pi rn : l$.

$\therefore$ $GF : dx=2\pi rn : l$. $\therefore$ $GF=(2\pi rn.dx)/l$.....(5).

(5) in (4) gives, $MK=[2\pi rn(l-x)dx]/l^2$.....(6).

(6) in (3) gives, $FD=[2\pi rnx(l-x)]/l^2$ .....(7).

(7) and (5) in (2) gives $PN=[4\pi^2r^2n^2(l-x)xdx]/rl^3$..(8).

The increment of $FD$ is $(GH+NE)=FG+NE)$.

$\therefore$ from (7) $d(FD)=[2\pi rn(l-2x)dx]/l^2=(FG+NE)$.(9).

(5) in (9) gives, $NE=[-4\pi rnxdx]/l^2$.....(10).

(8) and (10) in (1) gives,

$$ds=\frac{4\pi rnx}{l^2}\sqrt{1+\frac{\pi^2n^2}{l^2}(l-x)^2}dx. \quad \therefore s=\frac{4\pi rn}{l^2}\int_o^l\sqrt{1+\frac{\pi^2n^2}{l^2}(l-x)^2}xdx,$$

$$=(4r/3\pi n)+(2r/3)[\pi n-(2/\pi n)]\sqrt{1+\pi^2n^2}+2r\log(\pi n+\sqrt{1+\pi^2n^2}).$$

In the problem $r=1$, $n=100$.

$$\therefore s=(1/75\pi)+\tfrac{2}{3}[100\pi-(1/50\pi)]\sqrt{1+10000\pi^2}+2\log(100\pi+\sqrt{1+10000\pi^2}).$$

$\therefore$ $s=68948.771$ feet.

**66. Proposed by J. K. ELLWOOD, A. M., Principal of the Colfax School, Pittsburg, Pa.**

Around the top of a conical frustum,—base 5 feet, top 1 foot, altitude 100 feet,—is wound a rope 100 feet long and 1 inch thick. It is unwound by a hawk flying in one plane. How far does Mr. Hawk fly?

**Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.**

Instead of measuring the thickness of the rope on the slant height, in this solution it is measured on the altitude, the difference being for the whole winding not above .007 of an inch. Measuring from middle of rope to middle of rope, the first coil is $\frac{1}{2}$ inch below top of frustum, second coil $\frac{3}{2}$ inches, third $\frac{5}{2}$ inches, etc. Let $x$=height of cone.

$\therefore$ $x : \frac{5}{2}=x-100 : \frac{1}{2}$, $\therefore$ $x=125$.

$125-100=25$ feet, $=300$ inches.

$\therefore$ $300 : 6 = 300\frac{1}{2} : 6_{\frac{1}{100}}$,

$300 : 6 = 301\frac{1}{2} : 6_{\frac{3}{100}}$,

$300 : 6 = 302\frac{1}{2} : 6_{\frac{5}{100}}$.

. . . . . .

Let $n$=number of turns.

$\therefore$ $2\pi(6_{\frac{1}{100}}+6_{\frac{3}{100}}+6_{\frac{5}{100}}+\text{to } n \text{ terms})=1200$ inches.

$\therefore$ $2\pi(6+6+6+\ldots\ldots)+2\pi(_{\frac{1}{100}}+_{\frac{3}{100}}+_{\frac{5}{100}}+\ldots\ldots)=1200$.

$\therefore$ $12\pi n+\pi n^2/1200=100$. $\therefore$ $n=31.03$ coils. From the previous problem,

$$s=\frac{4\pi rm}{l^2}\int_{25}^{27\frac{7}{12}}\sqrt{1+\frac{\pi^2m^2}{l^2}(l-x)^2}.xdx, \text{ where } m=331,$$

the number of coils of rope to top of cone beginning with the bottom coil, and $l$=331 inches the height=$27\frac{7}{12}$ feet; also let 25 feet=$l'$, $r=\frac{331}{600}$ feet=radius of base of cone at bottom of rope.

$$\therefore\ s=\frac{4\pi rm}{l^2}\int_{l'}^{l}\sqrt{1+\frac{\pi^2m^2}{l^2}(l-x)^2}.xdx.$$

$$\therefore\ s=\frac{4r}{3\pi m}+\frac{2r}{3l^3\pi m}[(l^2+ll'-2l'^2)\pi^2m^2-2l^2]\sqrt{l^2+(l-l')^2\pi^2m^2}$$

$$+2r\log\left(\frac{(l-l')\pi m+\sqrt{l^2+(l-l')^2\pi^2m^2}}{l}\right).$$

$$s=(1/450\pi)+(1/900\pi)(28861\pi^2-2)\sqrt{1+961\pi^2}+\tfrac{331}{600}\log(31\pi+\sqrt{1+961\pi^2}).$$

$s$=9817.69235 feet.

In this solution the rope is unwound from the bottom.

## MECHANICS.

**55. Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University P. O., Miss.**

Three equal heavy spheres, each of weight $W$, are placed on a rough ground just not touching each other. A fourth sphere of weight $nW$ is placed on the top touching all three. Show that there is equilibrium if the coefficient of friction between two spheres is greater than $\tan\frac{1}{2}\alpha$, and that between a sphere and the ground is greater than $n\tan\frac{1}{2}\alpha/(n+3)$, where $\alpha$ is the inclination to the vertical of the straight line joining the centers of the upper and one lower sphere.

**Solution by the PROPOSER.**

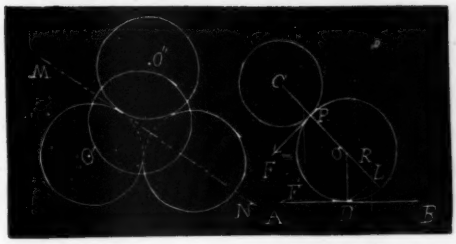
I.

In what different ways may motion occur, if the friction be such as to permit motion?

So long as the lower spheres maintain their positions, the upper will not move.

How, then, may the lower move? Referring to the section—

1. By sliding in the direction $BA$;
2. By rolling in the direction $BA$;
3. By rolling in the direction $AB$;
4. By sliding in the direction $AB$.




PLAN. SECTION MN.

Motions (1) and (2) cannot take place because, even if this did not at once bring the lower spheres into contact, such would necessarily raise the upper sphere and, evidently, there are no forces acting which could have this effect. Having disposed of (1) and (2), consider (3).

The forces acting on the sphere whose center is $O$ are its own weight, the friction $F'$, the friction $F$, and the normal action between the two spheres, $R$. Rolling in the direction $AB$ will not occur unless the moment of $R$ about $D$ is greater than that of $F$. Equating these two moments,

$$F.PL=R.DL,$$

$$F=(DL/PL)R$$

$$=\tan\tfrac{1}{2}\alpha.R,$$

since $\angle DPL=\frac{1}{2}\angle DOL$, and $\angle DOL=\alpha$, $OD$ being vertical and $OL$ the prolongation of $CO$.

If $\mu$=coefficient of friction between the two spheres, $F=\mu R$.

$\therefore\ \mu=\tan\frac{1}{2}\alpha$.

If, then, the coefficient of friction between the spheres whose centers are $C$ and 0 is greater than tan $\frac{1}{2}\alpha$, $F$ will prevent motion (3).

Now consider (4).

This will take place unless $F'$ is equal to, or greater than, $R\sin\alpha-F\cos\alpha$.

The normal pressure on the ground at $D$ is the weight of sphere $O$ plus one-third of the weight of sphere $C$, or $W+\frac{1}{3}nW$.

If $\mu'$ is the coefficient of friction between the sphere and the ground, $F'=\mu'(n+3)\frac{1}{3}W$, and, we have, for equilibrium,

$$\mu'(n+3)\tfrac{1}{3}W=R\sin\alpha-F\cos\alpha$$

$$=R(\sin\alpha-\tan\tfrac{1}{2}\alpha\cos\alpha),$$

taking the least value of $F$ consistent with equilibrium. From this,

$$R=\frac{\mu'(n+3)}{\tan\frac{1}{2}\alpha}\cdot\tfrac{1}{3}W.$$

Consider the forces acting on the upper sphere—its own weight $nW$, $F$, and $R$ reversed in direction, and, in addition, the actions $F$ and $R$ from each of the two other spheres, $O'$ and $O''$ (see plan). Resolving vertically,

$$nW=3R\cos\alpha+3F\sin\alpha$$
$$=3R(\cos\alpha+\tan\tfrac{1}{2}\alpha\sin\alpha)=3R;$$
$$R=\tfrac{1}{3}nW.$$

Equating the two values of $R$ and solving for $\mu'$,

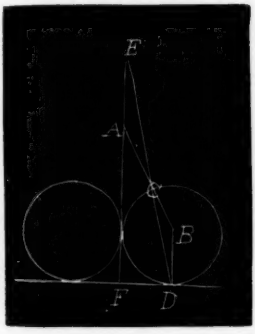
$$\mu'=\tan\tfrac{1}{2}\alpha.\frac{n}{n+3}.$$

REMARKS. Looking at the section it might seem, at a glance, that $R$ ought to bs $\frac{1}{3}nW\cos\alpha$, whereas it has been found to be $\frac{1}{3}nW$. It must be remembered that the two spheres, $C$ and $O$, are not in equilibrium by themselves, $C$ being held in position by reactions and friction from $O'$ and $O''$ as well as from $O$.

It may be noted, too, that the friction brought into play at $P$ may not be sufficient to prevent sliding. In order that motion of this kind may not take place the coefficient of friction must not be less than $\tan\alpha$. But the reason why $C$ does not slide is that it cannot descend unless $O$ rolls or slides out of its way toward $B$. And, as has been seen, neither of these motions can occur if $\mu>\tan\frac{1}{2}\alpha$ and $\mu'>\tan\frac{1}{2}\alpha.n/(n+3)$.

**III. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.**

Let $W$=weight of one of the equal spheres, $W_1$=weight of upper sphere, $\angle FAB=\alpha$.




Since sphere $B$ and reaction of the ground acts through $D$, the total resistance between spheres $A$ and $B$ must act through $D$.

Hence the total resistance between $A$ and $B$ acts in the line $ECD$.

$\therefore$ The angle of friction between $A$ and $B$ must be $\gtreqless$ than $\angle ACE=\frac{1}{2}\alpha$, $\therefore$ coefficient of friction $\gtreqless\tan\frac{1}{2}\alpha$.

Let $\mu$=coefficient of friction between $B$ and the ground, $R$=total normal, $P$=total tangential resistanee at $D$.

$\therefore\ \mu R=P$ or $\mu=P/R$.

$\therefore\ \mu=(\frac{1}{3}W_1\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha)/[(\frac{1}{3}W_1+W)\cos^2\frac{1}{2}\alpha]$.

$$\therefore\ \mu=\frac{W_1}{W_1+3W}\tan\tfrac{1}{2}\alpha=\frac{n}{n+3}\tan\tfrac{1}{2}\alpha,\ \text{when } W_1=nW.$$

Let $AC=b$, $CB=c$. Then $\sin\alpha=\dfrac{2c}{\sqrt{3}(b+c)}$.

$$\therefore\ \tan\tfrac{1}{2}\alpha=\frac{(b+c)\sqrt{3}-\sqrt{3b^2+6bc-c^2}}{2c}.$$

$\therefore\ \tan\frac{1}{2}\alpha=\sqrt{3}-\sqrt{2}$ when $b=c$.

## Note on Second Solution of Problem 49, Mechanics.

By ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University P. O., Miss.

My work as published in the June-July number is the solution of the following problem :

A uniform bar, whose other dimensions may be neglected in comparison with its length ($2a$), is placed in vertical position on the edge of a platform of given height and, being slightly displaced from the vertical, is caused to revolve about the edge as an axis, the lower end being set free when the bar becomes horizontal. Where and in what manner will it strike the ground ?

The discussion of either problem—the one just stated or the one actually proposed—naturally divides itself into two parts, the first dealing with the motion before, the second after, the stick leaves the platform. It was the latter that chiefly occupied my thoughts when I prepared the solution referred to above.

I now offer the following as a discussion of the first part :

Let $m$=mass of stick,

$2b$=length of stick,

$2a$=other dimension perpendicular to edge of platform,

$\theta$=angle through which stick has turned in $t$ seconds,

$\omega$=angular velocity at time $t$.

$$\frac{d^2\theta}{dt^2}=\frac{mgb\sin\theta}{m\{[(a^2+b^2)/3]+b^2\}}=\frac{3b\sin\theta}{a^2+4b^2}g;\ \omega^2=\left(\frac{d\theta}{dt}\right)^2=-\frac{6b\cos\theta}{a^2+4b^2}g+c,$$

or, since when $\theta=0$, $\omega=0$,

$$\omega^2=\frac{6bg}{a^2+4b^2}(1-\cos\theta.)\quad \text{Normal acceleration}=b\omega^2=\frac{6b^2g}{a^2+4b^2}(1-\cos\theta).$$

Component of acceleration due to gravity directly opposite to this=$g\cos\theta$.

At the instant these become equal the contact between stick and platform ceases.

This occurs when $\cos\theta=(6b^2)/(a^2+10b^2)$.

Up to this time (friction being supposed sufficiently great to prevent sliding) the stick has turned about the edge of the platform. After leaving the platform the center of gravity of the stick describes a parabola, and the stick revolves with a constant angular velocity, $\omega$, about its center of gravity. This part of the motion might be discussed as in the June-July number.

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## DIOPHANTINE ANALYSIS.

58. Proposed by E. S. LOOMIS, Ph. D., Professor of Mathematics in Cleveland West High School; Berea, O.

The base of a right-angled triangle is 105; find all the perpendiculars and hypotenuses to fit it, such that their values shall be integers.

I. Solution by the PROPOSER.

1. Let the hypotenuses and sides of a right-angled triangle be represented by $m^2+n^2$, $m^2-n^2$, and $2mn$.

2. Since in this problem we have to do with squares, we will draw a few observations from the squares of the integers, 1 to 18.

Int.: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18.
Sq.: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324.

3. Observations: (1) The difference between two consecutive squares is odd, the smallest difference being 3; therefore 105 is the difference between two consecutive squares; (2) The series of differences increases by 2; (3) Any difference is two times the number of the place of the lesser square in the series, $+1$, therefore let $x$=the number of the place of the lesser square; whence $2x+1=105$, or $x=52$, therefore all pairs of integers sought are confined to the series of squares of the integers 1 to 53; (4) The difference between squares in odd, or even, places is even; as 105, and all its factors, are odd, one must seek for differences of two squares, one occupying an odd, and the other occupying an even place.

4. Since $105=1\times105=3\times35=5\times21=7\times15=15\times7=21\times5=35\times3=105\times1$, there are eight cases to consider, each case giving 0, 1, or more than 1 pair of integers.

5. Since 105 is odd, we must have $m^2-n^2=105$.

6. Solution of cases: (1) Let $m^2-n^2=1\times105$, therefore $m=\sqrt{(105+n^2)}$. Since $n$ is less than $m$, values for $n$, by observation (3), will be integers less than 53. Therefore by inspection, when

$$\begin{aligned} n&=4,\quad 8,\quad 16,\quad 52,\\ m&=11,\quad 13,\quad 19,\quad 53,\\ m^2+n^2&=137,\quad 233,\quad 617,\quad 5313,\\ \text{and } 2mn&=88,\quad 208,\quad 608,\quad 5512. \end{aligned}$$

Therefore there are four pairs of integers, for $1\times105=(11^2-4^2)=(13^2-8^2)=(19^2-16^2)=(53^2-52^2)$.

(2) Let $3(m^2-n^2)=3\times35$; whence $m=\sqrt{(35+n^2)}$; by observation (3) $n<17$, and by inspection, $105=3\times35=3(6^2-1^2)=3(18^2-17^2)$. When $n=1$, 17; $m=6$, 18; $3(m^2+n^2)=111$, 1839; and $3(2mn)=36$, 1836, giving two pairs for $3\times35$.

(3) Let $105=5(m^2-n^2)=5\times21=5(5^2-2^2)=5(11^2-10^2)$; whence $m=\sqrt{(21+n^2)}$. When $n=2$, 10; $m=5$, 11; $5(m^2+n^2)=145$, 1105; and $5(2mn)=100$, 1100, giving two pairs.

(4) Let $105=7\times15=7(m^2-n^2)=7(4^2-1^2)=7(8^2-7^2)$; whence $m=\sqrt{(15+n^2)}$. Therefore $7(m^2+n^2)=119$, 791; and $7(2mn)=56$, 784, giving two pairs.

(5) Let $105=15\times7=15(m^2-n^2)=15(4^2-3^2)$; whence $m=\sqrt{(7+n^2)}$, and $15(m^2+n^2)=375$, and $15(2mn)=360$, giving one pair.

(6) Let $105=21\times5=21(m^2-n^2)=21(3^2-2^2)$; whence $m=\sqrt{(5+n^2)}$, and $21(m^2+n^2)=273$, and $21(2mn)=252$, giving one pair.

(7) Let $105=35\times3=35(m^2-n^2)=35(2^2-1^2)$; whence $m=\sqrt{(3+n^2)}$, and $35(m^2+n^2)=175$, and $35(2mn)=140$, giving one pair.

(8) Let $105=105\times1=105(m^2-n^2)$. But 1 can not be the difference between two squares, by observation (1). Hence for $105=105\times1$ there is no solu-

tion. Therefore there can be but 13 pairs of integers which will satisfy the conditions of the problem, and the hypotenuses are 137, 233, 617, 5513, 111, 1839, 145, 1105, 119, 791, 375, 273, and 175; and the perpendiculars are 88, 208, 5512, 36, 1836, 100, 1100, 56, 784, 360, 252, and 140.

The principles established and employed in the above problem will hold whenever the given number is odd. But if the number is even, say 104, then from the series of squares above new observations must be drawn and employed.

II. Solution by SYLVESTER ROBBINS, North Branch, New Jersey.

Every rational right-angled triangle having 105 for its base must have an even number for its perpendicular and an odd one for its hypotenuse. Notice that of the factors entering into 105, each of these, 3, 5, 7, is difference of two squares *once*; each of the factors, 15, 21, 35, is the difference of two squares *twice*; and 105 itself is the difference of two squares *four times*. The perpendicular in each of these triangles must be twice the *product* of two roots, the base must be the difference of their squares, and the hypotenuse must be the *sum* of the same two squares. In order to obtain the *greatest* number of rational right triangles having same base, the $n$ factors of said base must be prime numbers, never using 2, then $(3^n-1)/2$ will express the number of triangles, 1, 4, 13, 40, 121, 364, 1093, 3280, etc. In this problem there are $3\times1+3\times2+1\times4=13$ triangles. The following are general expressions for sides of rational right triangles. Roots: $r=2\sqrt{35}$, $3\sqrt{21}$, $4\sqrt{15}$, $6\sqrt{3}$, $18\sqrt{3}$, $5\sqrt{5}$, $11\sqrt{5}$, $4\sqrt{7}$, $8\sqrt{7}$, 11, 13, 19, 53; $s=1\sqrt{35}$, $2\sqrt{21}$, $3\sqrt{15}$, $1\sqrt{3}$, $17\sqrt{3}$, $2\sqrt{5}$, $10\sqrt{5}$, $1\sqrt{7}$, $7\sqrt{7}$, 4, 8, 16, 52; base $=r^2-s^2=105$ in each case; perpendiculars $=2rs=140$, 252, 360, 36, 1836, 100, 1100, 56, 784, 88, 208, 608, 5512; $h=r^2+s^2=175$, 273, 375, 111, 1839, 145, 1105, 119, 791, 137, 233, 617, 5513, being the 13 values in order, respectively.

III. Solution by JOSIAH H. DRUMMOND; LL. D., Portland, Me., and G. B. M. ZERR, A. M., Ph. D., The Russell College, Lebanon, Va.

Let $h$=hypotenuse, $p$=perpendicular. Then $h^2-p^2=(105)^2$ or $h+p=(105)^2/(h-p)$. $\therefore$ $h-p$ must be some factor of $(105)^2$; $h+p$ is also; and as the factors are all odd, we have integral values of $h$ and $p$ for every integral factor, and, moreover, for all factors less than 105, these values will be positive. The prime factors are 1, 3, 3, 5, 5, 7, 7; and the factors less than 105, of $h-p$, are readily found to be 1, 3, 5, 7, 9, 13, 21, 25, 35, 45, 49, 63, 75; the corresponding factors, or values of $h+p$ are 11025, 3675, 2205, 1575, 1225, 735, 525, 441, 315, 245, 225, 175, 147. Hence, $h=5513$, 1839, 1105, 791, 617, 375, 273, 233, 175, 145, 137, 119, 111; and $p=5512$, 1836, 1100, 784, 608, 360, 252, 208, 140, 100, 88, 56, 36; and $b=105$ in each case.

M. A. GRUBER, A. M., War Department, Washington, D. C., finds the 13 sets of values as given above, and refers to the full explanation of finding these results as given in his paper on "Integral Sides of Right Triangles," pages 106—108, of Vol. IV, No. 4 (April, 1897), of MONTHLY, where the problem, "Given one of the legs of a right-triangle of integral sides to find the other leg and the hypotenuse," is fully treated.

A. H. BELL, Hillsboro, Ill., finds the 13 sets of values, and deduces the general rule for finding the same from the principle shown in his solution of Problem 31, page 363, Vol. II, of MONTHLY. P. S. BERG, A. M., Principal of Schools, Larimore, N. D., sends a good solution finding the 13 sets of values by the method of solution No. I, of the problem last referred to.

WILLIAM HOOVER, A. M., Ph. D., Ohio University, Athens, Ohio, solves substantially as in Solution I, above, but as he did not carry out the work far enough to give all the results, we adopted the phraseology of the Proposer.

Also excellently solved, but without determining the full sets of results, by OTTO CLAYTON, A. B., Remington, Ind.; J. O. MAHONEY, B. E., M. Sc,. Lynnville, Tenn.; and O. W. ANTHONY, M. Sc., Instructor in Mathematics in Boys' High School, New York City.

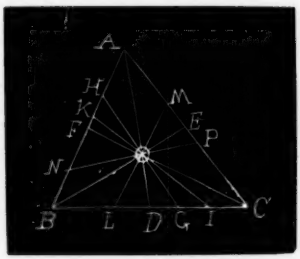
## AVERAGE AND PROBABILITY.

**56. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.**

Find the chance that the center of gravity of a triangle lies inside the triangle formed by three points taken at random within the triangle. [From *Williamson's Integral Calculus.*]

**Solution by HENRY HEATON, M. Sc., Atlantic, Iowa.**

Let $D$, $E$, and $F$ be the middle points of the respective sides of the triangle. Then $O$, the common intersection of $HD$, $BE$, and $CF$, is the center of gravity of the triangle. Suppose that the first point is somewhere upon the line $OI$, and the second somewhere upon the line $OG$. Put $DI=x$ and $DG=y$. Then the favorable positions for the third point are upon the surface $OKH$, whose area is




$$\frac{\triangle}{3}\left(\frac{4x}{a+6x}-\frac{4y}{a+9y}\right)$$

$y$ being less than $x$, and $x$ less than $\frac{1}{2}a$.

If $LD=y$, and the second point be upon $OL$, the favorable positions for the third point are upon the surface $OKAMO$, whose area is

$$\frac{\triangle}{3}\left(\frac{4x}{a+6x}+\frac{4y}{a+6y}\right).$$

If $FN=y$ and the second point be upon the line $ON$, the favorable positions for the third point are upon the surface $OKAPO$, whose area is

$$\frac{\triangle}{3}\left(\frac{4x}{a+6x}+1-\frac{4y}{a+6y}\right).$$

If $G$ be taken between $I$ and $C$, $H$ will fall between $F$ and $K$, and the favorable positions for the third point will be upon $OHK$, whose area is

$$\frac{\triangle}{3}\left(\frac{4y}{a+6y}-\frac{4x}{a+6x}\right).$$

The chance that the first point falls upon the element of surface at $OI$ is $(\triangle/3)(dx/a)/\triangle=dx/3a$.

The chance that the second falls upon the element of surface at $OG$ or $OL$ is $dy/3a$. The chance that it falls upon the element at $ON$ is $dy/3c$.

The chance that the third point falls upon any surface $S$ is $S/\triangle$.

Hence if the first point be confined to the surface $DOC$ and the second to surface $CFB$, the required chance is

$$P_1=\frac{1}{27a^2}\left[\int_0^{\frac{1}{2}a}\int_0^{x}\left(\frac{4x}{a+6x}-\frac{4y}{a+6y}\right)dxdy+\int_0^{\frac{1}{2}a}\int_0^{y}\left(\frac{4y}{a+6y}-\frac{4x}{a+6x}\right)dydx\right.$$
$$\left.+\int_0^{\frac{1}{2}a}\int_0^{\frac{1}{2}a}\left(\frac{4x}{a+6x}+\frac{4y}{a+6y}\right)dxdy\right]+\frac{1}{27ac}\int_0^{\frac{1}{2}a}\int_0^{\frac{1}{2}c}\left(\frac{4x}{a+6x}+1-\frac{4y}{c+6y}\right)dxdy.$$

By combining symmetrical expressions this reduces to

$$P_1=\frac{8}{27a^2}\left[\int_0^{\frac{1}{2}a}\int_0^{x}\left(\frac{x}{a+6x}\right)dxdy-{}^{*}\int_0^{\frac{1}{2}a}\int_0^{y}\left(\frac{x}{a+6x}\right)dydx\right.$$
$$\left.+\int_0^{\frac{1}{2}a}\int_0^{\frac{1}{2}a}\left(\frac{x}{a+6x}\right)dxdy\right]+\frac{4}{27ac}\left[\int_0^{\frac{1}{2}a}\int_0^{\frac{1}{2}c}\left(\frac{x}{a+6x}\right)dxdy\right.$$
$$\left.-\int_0^{\frac{1}{2}c}\int_0^{\frac{1}{2}a}\left(\frac{y}{c+6y}\right)dydx+\tfrac{1}{4}\int_0^{\frac{1}{2}a}\int_0^{\frac{1}{2}c}dxdy\right]$$
$$=\frac{8}{27a^2}\left[\int_0^{\frac{1}{2}a}\left(\frac{x^2}{a+6x}\right)-\frac{x(\frac{1}{2}a-x)}{a+6x}+\tfrac{1}{2}a\left(\frac{x}{a+6x}\right)\right]$$
$$+\frac{4}{27ac}\left[\int_0^{\frac{1}{2}a}\tfrac{1}{2}c\left(\frac{x}{a+6x}\right)dx-\int_0^{\frac{1}{2}c}\tfrac{1}{2}a\left(\frac{y}{c+6y}\right)dy+\tfrac{1}{2}c\int_0^{\frac{1}{2}a}dx\right]$$

$=\frac{1}{9}\frac{3}{7}\frac{}{2}+\frac{4}{7}\frac{}{9}\log 2$. Since this is independent of $a$, $b$, and $c$, it is evident that if the second point had been allowed to occupy all positions on both sides of $CF$, the result would have been twice as great, and if the first point had been allowed to take all positions instead of being confined to $DOC$ the result would have been six times as great.

Hence the required probability is $P=12P_1=\frac{1}{8}\frac{3}{1}+\frac{1}{2}\frac{6}{4}\frac{}{3}\log 2=\frac{1}{8}\frac{}{1}[13+\frac{1}{3}^{6}\log 2]$ $=.20613$.

REMARK. In integrating the expression marked *, I reversed the order of integration. Thus,

$$\int_0^{\frac{1}{2}a}\int_0^{y}\left(\frac{x}{a+6x}\right)dydx=\int_0^{\frac{1}{2}a}\int_x^{\frac{1}{2}a}\left(\frac{x}{a+6x}\right)dxdy=\int_0^{\frac{1}{2}a}\left(\frac{x(\frac{1}{2}a-x)}{a+6x}\right)da.$$

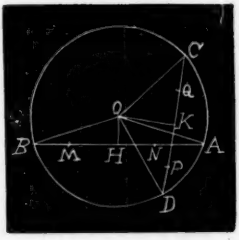
[The result given in *Williamson's Calculus* is $\frac{1}{2}\frac{}{7}(2+\frac{1}{3}^{0}\log 4$. EDITOR.]

57. Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

A chord is drawn through two points taken at random in the surface of a circle. If a second chord be drawn through two other points taken at random in the surface, find the chance that the quadrilateral formed by joining the extremities of the two chords will contain the center of the circle.

I. Solution by the PROPOSER.

Let $M, N$ be the first two random points; $AB$ the chord through them; $P$, $Q$ the second two random points; $C$, $D$ the chord through them; and $O$ the center of the circle. Draw $OH$, $OK$ perpendicular to $AB$, $CD$.




Let $AO=r$, $AM=w$, $MN=x$, $CP=y$, $PQ=z$, $\angle AOH=\theta$, $\angle COK=\varphi$, $\angle AOC=\psi$, and $\mu$=the angle $AB$ makes with some fixed line.

Then $AH=r\sin\theta$, $CK=r\sin\varphi$; an element of the circle at $M$ is $r\sin\theta d\theta dw$; at $N$, $d\mu x dx$; at $P$, $r\sin\varphi d\varphi dy$; at $Q$, $d\psi z dz$. The limits of $\theta$ are 0 and $\frac{1}{2}\pi$; of $\varphi$, 0 and $\theta$, and doubled; of $\psi$, $\pi-2\theta$ and $\pi$, and doubled; of $\mu$, 0 and $2\pi$; of $w$, 0 and $2r\sin\theta=s$; of $x$, 0 and $w$, and doubled; of $y$, 0 and $2r\sin\varphi=v$; and of $z$, 0 and $y$, and doubled. Hence, since the whole number of ways the four points can be taken is $\pi^4r^8$, the required chance is

$$p=\frac{16}{\pi^4r^8}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\pi-2\theta}^{\pi}\int_0^{2\pi}\int_0^{s}\int_0^{w}\int_0^{v}\int_0^{y} r\sin\theta d\theta r\sin\varphi d\varphi d\psi d\mu dw x dx dy z dz$$

$$=\frac{8}{\pi^4r^6}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\pi-2\theta}^{\pi}\int_0^{2\pi}\int_0^{s}\int_0^{w}\int_0^{v}\sin\theta\sin\varphi d\theta d\varphi d\psi d\mu dw x dx y^2 dy$$

$$=\frac{64}{3\pi^4r^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\pi-2\theta}^{\pi}\int_0^{2\pi}\int_0^{s}\int_0^{w}\sin\theta\sin^4\varphi d\theta d\varphi d\psi d\mu dw x dx$$

$$=\frac{32}{3\pi^4r^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\pi-2\theta}^{\pi}\int_0^{2\pi}\int_0^{s}\sin\theta\sin^4\varphi d\theta d\varphi d\psi d\mu w^2 dw$$

$$=\frac{256}{9\pi^4}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\pi-2\theta}^{\pi}\int_0^{2\pi}\sin^4\theta\sin^4\varphi d\theta d\varphi d\psi d\mu$$

$$=\frac{512}{9\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\pi-2\theta}^{\pi}\sin^4\theta\sin^4\varphi d\theta d\varphi d\psi=\frac{1024}{9\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\theta\sin^4\theta\sin^4\varphi d\theta d\varphi$$

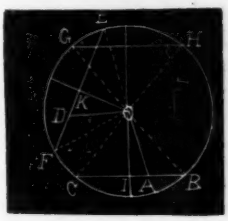
$$=\frac{128}{9\pi^3}\int_0^{\frac{1}{2}\pi}(3\theta^2-3\theta\sin\theta\cos\theta-2\theta\sin^3\theta\cos\theta)\sin^4\theta d\theta=\frac{2}{3}+\frac{139}{72\pi^2}.$$

II. Solution by HENRY HEATON, M. Sc., Atlantic, Iowa.

Let $A$ and $D$ represent the positions of the first and third points, and sup-

pose the second and fourth to be taken somewhere upon the lines $BC$ and $EF$, respectively.

Put $OC=a$, $OA=x$, $OD=y$, $\angle OAC=\theta$, $\angle IOC=\phi$, $\angle ODK=\psi$, and $\angle KOF=\omega$.




Then $a\cos\phi=x\sin\theta$, and $a\cos\omega=y\sin\psi$.

If either $E$ or $F$ is upon the arc $HG$, the quadrilateral formed by joining the extremities of the chords will contain the center of the circle.

Since the arc $HG$=the arc $CB$, the probability of this is $(\phi+\omega)/\pi$.

If $x>0$ and $<a$ the chance that the first point is taken between $x$ and $x+dx$ is $2\pi x dx/a^2\pi=2xdx/a^2$.

If $y>0$ and $<a$ the chance that the third point is taken between the distances $y$ and $y+dy$ is $2ydy/a^2$.

If $\theta>0$ and $<\frac{1}{2}\pi$ the chance that the second point is taken between the line $BC$ and a second line making at $A$ the angle $d\theta$ is $\frac{1}{2}(AC^2+AB^2)d\theta/\frac{1}{2}a^2\pi$ $=2[a^2+x^2(1-2\sin^2\theta)]d\theta/a^2\pi=2(1-2\cos^2\phi+\cos^2\phi\operatorname{cosec}^2\theta)d\theta/\pi$.

If $\psi>0$ and $<\frac{1}{2}\pi$ the chance that the fourth point is taken between the line $EF$ and a second line making at $D$ the angle $d\psi$ is $2(1-2\cos^2\omega+\cos^2\omega\operatorname{cosec}^2$ $\psi)d\psi/\pi$.

If we suppose $\theta$ constant while $x$ and $\phi$ vary, $xdx=-a^2\sin\phi\cos\phi\operatorname{cosec}^2\theta d\phi$. When $x=0$, $\phi=\frac{1}{2}\pi$, and when $x=a$, $\phi=\frac{1}{2}\pi-\theta$. Hence the limits of integration for $\phi$ are $\frac{1}{2}\pi-\theta$ and $\frac{1}{2}\pi$. If we integrate first with respect to $\theta$, the limits for $\theta$ are $\frac{1}{2}\pi-\phi$ and $\frac{1}{2}\pi$, while for $\phi$ they are 0 and $\frac{1}{2}\pi$.

In like manner we may substitute $-a^2\sin\omega\cos\omega\operatorname{cosec}^2\psi d\omega$ for $ydy$ and integrate between limits $\frac{1}{2}\pi-\omega$ and $\frac{1}{2}\pi$ for $\psi$, and between 0 and $\frac{1}{2}\pi$ for $\omega$.

Hence the required probability is

$$P=\frac{16}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi-\phi}^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi-\omega}^{\frac{1}{2}\pi}(\phi+\omega)(1-2\cos^2\phi+\cos^2\phi\operatorname{cosec}^2\theta)\times$$

$$(1-2\cos^2\omega+\cos^2\omega\operatorname{cosec}^2\psi)\sin\phi\cos\phi\operatorname{cosec}^2\theta\sin\omega\cos\omega\operatorname{cosec}^2\psi d\phi d\omega d\theta d\psi$$

$$=\frac{256}{9\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}(\phi+\omega)\sin^4\phi\sin^4\omega d\phi d\omega$$

$$=\frac{4}{9\pi^3}\int_0^{\frac{1}{2}\pi}(12\pi\phi+3\pi^2+16)\sin^4\phi d\phi=\tfrac{1}{2}+(8/3\pi^2)=.77019.$$

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## MISCELLANEOUS.

**55. Proposed by J. M. COLAW, A. M., Monterey, Va.**

Multiply 6 by 4. Is the problem legitimate when both symbols represent pure number?

[NOTE. "A measured or numbered quantity may be divided into a number of parts, or taken a number of times; but no number can be multiplied or divided into parts."—*McLellan and Dewey's Psychology of Number*. "The astounding thesis is maintained that number is not a magnitude, does not possess quantity at all, and that 'no number can be multiplied or divided into parts'."—*Lefevre's Number and Its Algebra*.]

I. Comment by DAVID EUGENE SMITH, Ph. D., Professor of Mathematics, Michigan State Normal School, Ypsilanti, Mich.

This question seems to me to have no interest to mathematicians. It simply means that somebody has set up a narrow definition of multiplication and has then said, "*Your* work is not multiplication because it does not fit *my* definition." I expressed my humble opinion in the MONTHLY some time ago when the antiquated definition of division was brought up to prove that it was impossible to divide $10 by 2. Such narrow limitations seem to me utterly nonsensical.

In a similar sense we cannot multiply by $-1$, and we cannot have "imaginary numbers," and 1 is not a number, etc., etc. Mathematical progress has always been made the more difficult because somebody has insisted on hanging on to some antiquated definition.

What do these people who say that we cannot multiply 2 by 3 say to some such simple formula as $e^{\pi i}=-1$ ? I suppose they say that $e$, $\pi$, $i$, and $-1$ have no existence.

II. Comment by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

There may be a modern idea that since neither 6 nor 4 is a numbered quantity, the operation is impossible. If it were possible to get the evidence of all the mathematicians, I am sure not one could be found who did not learn his multiplication table, in fact, get his aptness in numbers by the same process as given in the problem. There may be some who claim otherwise but I would even doubt their claim.

$6\times 4=24$ is good arithmetic. It seems a pity that vandals should make incursions upon the sacred shrines of Newton, La Place, Pierce, and other noted men of numbers, and so desecrate their immortal works, as to try and mistify their teachings. The God of Mathematics will not permit it.

III. Remarks by J. K. ELLWOOD, A. M., Principal of the Colfax School, Pittsburg, Pa.

Do $4\times 6=24$ ? Can 6 be multiplied by 4 ? Six what ? If 6 units of quantity, yes; but if not a magnitude,—well, what then is it ? "Six" apart from the universe of space, time, and matter, suggests to the mind—what ? The "how many"? The *ratio* of the "how many" to the unit ? Six in the abstract —a pure number—can not, in an arithmetical sense, be multiplied by any abstraction. In an algebraic sense, $4\times 6=24$, just as $x\times x=x^2$. That is, we operate with *symbols*, neglecting the realities represented. If two abstract numbers can be multiplied one by the other, why not two concrete numbers, as feet $\times$ feet =square feet ?

42. Proposed by E. B. ESCOTT, Cambridge, Mass.

To find triangles whose sides and median lines are commensurable.

II. Solutions communicated to "L' Intermediaire des Mathematiciens" (January, 1898) by the PROPOSER. Selected and translated by J. M. COLAW, A. M., Monterey, Va.

*First Solution.* By Chas. Gill (New York, 1848).

$x=t[1-(\cos A+\sin A)(\cos B+\sin B)]$, $y=t[\cos B-\sin B+(\cos A-\sin A)(\cos B+\sin B)]$, $z=t[\cos A-\sin A+(\cos B-\sin B)(\cos A+\sin A)$, whence there exists one of the four relations following:

$\tan\frac{1}{2}A=(16+13\sin B-8\cos B-5\sin 2B-2\sin^2 B)/[(2+\sin B+2\cos B)(5+4\sin B-4\cos B)]$.....(I) ; $\tan\frac{1}{2}A=[(1-\sin B)(5-\cos B+4\sin B)]/[(\sin B+\cos B)(1+3\cos B-\sin B)]$.....(II) ; $\cot\frac{1}{2}A=-(16-13\cos B+8\sin B-5\sin 2B-2\cos^2 B)/[(2-\cos B-2\sin B)(5+4\sin B-4\cos B)$.....(III) ; $\cot\frac{1}{2}A=[(1+\cos B)(5+\sin B-4\cos B)]/[(\sin B+\cos B)(1+\cos B-3\sin B)]$.....(IV). Letting, in formula (II), $\sin B=\frac{4}{5}$, $\cos B=-\frac{3}{5}$, then $\tan\frac{1}{2}A=-\frac{1}{2}$, $x=262$, $y=316$, $z=254$, $m_x=261$, $m_y=204$, and $m_z=255$.

*Second Solution.* From *The Gentleman's Mathematical Companion*, London, 1824, page 350.

Let $z=x+y-d$, then $4m_z^2=x^2-2xy+y^2+2d(x+y)-d^2$; $4m_y^2=4x^2+4xy+y^2-4d(x+y)+2d^2$; $4m_x^2=x^2+4xy+4y^2-4d(x+y)+2d^2$.

Let $4x^2+4xy+y^2-4d(x+y)+2d^2=(2x+y-m)^2$; then $x=(2d^2-4dy+2my-m^2)/(4d-4m)$, $y=(2d^2-4dx+4mx-m^2)/(4d-2m)$.

Let $x^2+4xy+4y^2-4d(x+y)+2d^2=(x+2y-n)^2$; then $x=(2d^2-4dy+2ny-n^2)/(4d-2n)$ ; $y=(2d^2-4dx+2nx-n^2)/(4d-4n)$.

$x=[d^2(4n-2m)+2d(m^2-n^2)-mn(2m-n)]/[4d(m+n)-6mn]$ ;

$y=[d^2(4m-2n)-2d(m^2-n^2)-mn)2n-m)]/[4d(m+n)-6mn]$;

$z=x+y-d=-[2d^2(m+n)-6mnd+mn(m+n)]/[4d(m+n)-6mn]$.

We may neglect the common denominator $4d(m+n)-6mn$. We then have to satisfy the condition, $2x^2+2y^2-z^2=36d^4(m-n)^2-24d^3(m+n)(2m^2-5mn+2n^2)+4d^2(4m^4+7m^2n-39m^2n^2+7mn^3+4n^4)-12dmn(m+n)(2m^2-5mn+2n^2)+9m^2n^2(m-n)^2=$a square.

We have also $=\{6d^2(m-n)-2d[(m+n)/(m-n)](2m^2-5mn+2n^2)-3mn(m-n)\}^2$.

Whence $d=[3(m+n)(m-n)^2]/(5m^2-8mn+5n^2)$.

Letting $m=3$, $n=2$, $d=1\frac{5}{7}$, $x=656$, $y=414$, $z=290$,
$m_x=142$, $m_y=463$, $m_z=529$,

$m=3$, $n=1$, $d=2\frac{4}{13}$, $x=174$, $y=170$, $z=136$,
$m_x=127$, $m_y=131$, $m_z=158$,

$m=3$, $n=-1$, $d=4\frac{8}{37}$, $x=650$, $y=318$, $z=628$,
$m_x=377$, $m_y=619$, $m_z=404$,

$m=5$, $n=4$, $d=\frac{3}{5}$, $x=892$, $y=554$, $z=954$,
$m_x=640$, $m_y=881$, $m_z=569$.

[*E. Fauquembergue* says, (*L' Intermediaire*, *Mars* 1897), that Euler, at different times busied himself with the problem of finding a triangle whose sides and medians are commensurable. His solution is reproduced in the *Commentationes Arithmeticæ collectæ*, *t.* II, page 488. He gives formulæ from which may be obtained an indefinite number of triangles answering the conditions of the problem. He deduces, among others, the solution with the integers 68, 85, 87 for the sides. See pages 94-95, of MONTHLY, Vol. IV, 1897, for solution I, by M. Tesch. EDITOR.]

# PROBLEMS FOR SOLUTION.

## ARITHMETIC.

90. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Find the greatest number of inch balls that can be placed in a box 10 inches square and 5 inches deep.

91. Proposed by RAYMOND D. SMITH, Tiffin, Ohio.

A barn 20 feet square is standing in a pasture, and a horse is tied to one corner of it with a rope 50 feet long. Over how much land can he graze ?

92. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pa.

What rate of income do I realize by purchasing United States 4% bonds at 105 if I sell them in six years at 104 ?

*** Solutions of these problems should be sent to B. F. Finkel, not later than April 10.

## GEOMETRY.

88. Proposed by FREDERICK R. HONEY, Ph. B., Instructor in Mathematics in Trinity College, New Haven, Conn.

Prove that the volume of the frustum of a cone is equal to one-sixth of the altitude multiplied by the sum of the areas of the upper base, the lower base, and four times the area of the section midway between the upper and lower bases.

89. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Describe a circle tangent to three given circles. [From *Chauvenett's Geometry*, page 318, ex. 213.]

90. Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

The bisectors of the angles of the opposite sides (produced) of an inscribed quadrilateral cut the sides at the angular points of a rhombus.

*** Solutions of these problems should be sent to B. F. Finkel, not later than April 10.

## CALCULUS.

70. Proposed by J. OWEN MAHONEY, B. E., M. Sc., Graduate Fellow in Mathematics in Vanderbilt University, P. O., Lynnville, Tenn.

Prove $\int_0^\infty \frac{\cos ax}{1+x^{2n}}dx = -i\frac{\pi}{n}\sum_{r=1}^{n}\omega^{2r-1}e^{ai\omega^{2r-1}}$

where $n$ is an integer, $a$ is positive, and $\omega$ is $e^{i\pi/2n}$.

Is this correct ? Forsyth gives, on page 41, of his *Theory of Functions*, the integral

$$\int_{-\infty}^{\infty}\frac{\cos axdx}{1+x^{2n}} = -i\frac{\pi}{2l}\sum_{r=1}^{n}\omega^{2r-1}e^{ai\omega^{2r-1}}.$$

71. Proposed by J. C. CORBIN, Pine Bluff, Ark.

Form the differential equation of the third order, of which

$$y=c_1e^{2x}+c_2e^{-3x}+c_3e^{x}$$ is the complete primitive.

72. Proposed by G. B. M. ZERR, A. M.,Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

A man has a park in the form of a parabolic segment cut off by a chord making an angle $\pi/4$ with the axis. Within the park is a right angled triangular flower plat with one vertex at the center of gravity of the segment, the other vertex at the lower extremity of the chord, and the right angle on the diameter bisecting the chord. The park contains 30 acres, and the perimeter of triangle in linear measure equals the area in square measure. Find the length of the chord, the latus-rectum of the parabola, and the dimensions of the triangle.

*** Solutions of these problems should be sent to J. M. Colaw, not later than April 10.

## MECHANICS.

64. Proposed by B. F. FINKLE, A. M.,M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A cylindrical vessel, radius of vessel $r$ and altitude $h$, is filled with water and rests on a horizontal plane. It is required to ascertain the maximum angle of elevation to which the plane may be raised without the vessel falling, allowing the coefficient of friction to be such as to prevent sliding, and the water to overflow as the plane is raised.

65. Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

The distance, parallel to the axis, from the mid-point of a chord to the arc of a parabola is constant. Show that the center of gravity of all segments formed by the chord is an equal parabola.

*** Solutions of these problems should be sent to B. F. Finkel, not later than April 10.

## DIOPHANTINE ANALYSIS.

62. Proposed by JOHN M. ARNOLD, Crompton, R. I.

Find, if possible, four square numbers in arithmetical progression.

63. Proposed by A. H. HOLMES, Brunswick, Me.

Given $x^2+y^3=20^3\times 105498$, to find four positive *integral* values each for $x$ and $y$.

64. Proposed by SYLVESTER ROBINS, North Branch Depot, N. J.

It is required to take from the proper *key* suitable material and hastily construct a "nest" of 10 or 15 prime, integral, rational trapeziums, each containing an area equal to the square root of the product of its four sides.

65. Proposed by MANSFIELD MERRIMAN, Professor of Civil Engineering, Lehigh University, South Bethlehem, Pa.

Show that the number 1521 can be expressed in seven different ways as the sum of three perfect squares? Can more than seven different ways be found?

*** Solutions of these problems should be sent to J. M. Colaw, not later than April 10.

## AVERAGE AND PROBABILITY.

61. Proposed by COL. CLARKE.

A cube being cut at random by a plane, what is the chance that the section is a hexagon? [From *Williamson's Integral Calculus.*]

62. Proposed by O. S. KIBLER, Superintendent of Schools, Middleburg, O.

A bag contains any number of balls, which are equally likely to be white or black; one is drawn and found to be white. Show that the chance of drawing another white one, the first ball not being replaced, is two-thirds. [From *C. Smith's Treatise on Algebra*, page 615.]

*** Solutions of these problems should be sent to B. F. Finkel, not later than April 10.

## MISCELLANEOUS.

60. Proposed by S. HART WRIGHT, M. D., A. M., Ph. D., Penn Yan, N. Y.

When the Sun's declination is 23° 27′ 15″ North=$\delta$, in what latitude will it shine on the *north* side of buildings during the first half of the forenoon, and on the *south* side during the other half, and what will be the length of the day?

61. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

The product of $n$ numbers, each the sum of four squares, may be expressed as the sum of four squares in $(48)^{n-1}$ different ways.

*** Solutions of these problems should be sent to J. M. Colaw, not later than April 10.

# NOTES.

## THE EVANSTON MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The second meeting of the Chicago Section of the American Mathematical Society was held at Evanston, Ill., on December 30th and 31st, 1897. The first session was called to order in the morning of December 30th by the chairman of the section, Head Professor Moore, University of Chicago. The following list of papers was announced:

1. Independent computation of integrals involving the square root of a quadric or cubic expression. Professor Henry Benner, Albion College.
2. Upon a ruled surface of the fourth order mechanically generated. Dr. E. M. Blake, Purdue University.
3. Upon articulated systems. Dr. E. M. Blake, Purdue University.
4. On the cubic involution and the theory of elliptic functions. Professor Oskar Bolza, University of Chicago.
5. Approximate solution of a particular differential equation. Dr. James H. Boyd, University of Chicago.
6. A note on reticulations. Professor Ellery W. Davis, University of Nebraska.

7. On systems of curves depending upon a single parameter. Dr. L. W. Dowling, University of Wisconsin.
8. A generalized Legendre's coefficient. Dr. James W. Glover, University of Michigan.
9. On a geometrical papyrus of the first century. Mr. Edgar J. Goodspeed, University of Chicago.
10. Alternate processes. Professor Arthur S. Hathaway, Rose Polytechnic Institute.
11. On a remarkable class of hyperspherical tetrahedra. Professor C. H. Hinton, University of Minnesota.
12. Multiple totients. Mr. D. N. Lehmer, University of Nebraska.
13. The determination of all ternary and quaternary symmetrical and alternating collineation-groups. Professor H. Maschke, University of Chicago.
14. On the solvability of groups. Dr. G. A. Miller, Cornell University.
15. Concerning the general equations of the seventh and eighth degrees. Professor E. H. Moore, University of Chicago.
16. Continuous Groups of Spherical Transformations. Professor H. B. Newson, University of Kansas.
17. Normal forms of projective transformations. Professor H. B. Newson, University of Kansas.
18. Cantor's transfinite numbers. Professor James Byrnie Shaw, Illinois College.
19. A new kind of number. Professor James Byrnie Shaw, Illinois College.
20. Twisted quartic curves of the first species and certain associated quartics. Professor Henry S. White, Northwestern University.

Professor Moore stated that a paper from Dr. Dickson, California University, reached him too late to be placed in the list. On account of the richness of the list of papers it was necessary to request the authors to be as brief as possible in presenting them. Regardless of these efforts to save time, the sessions had to be very long in order to get through with the work in two days.

As a whole the papers were of a high grade and furnished another index that we are moving rapidly towards higher types of mathematical activity. It is to be hoped that this forward movement will keep pace with the improvements in our library facilities. Through the generosity of some of our largest libraries it is becoming possible to keep in fairly close touch with the progress in our science, even if one is removed from the library centers.

One of the most pleasant features of the meeting was the fact that nearly all the authors of the papers were present and could thus present their own papers. It was also encouraging to see a number of mathematicians from a distance who were not represented on the program. Among these we may mention Professor Ziwet, editor of the "Bulletin" of the society; Professor Markley, Michigan University; Professor Skinner, Wisconsis University; Professor Waldo, DePaw University; Mr. A. C. Burnham, Illinois University; Professor Winston, Kansas Agricultural College, etc.

In our country of great distances such meetings seem especially important, not only on account of the inspiration resulting from the close contact with those of similar pursuits, but also on account of the opportunities to correct erroneous impressions in regard to the relative importance of the various lines of investigations. It is to be hoped that the future meetings of the Chicago Section will exert a still stronger influence on Western mathematics in both of these lines.

G. A. MILLER.

*Ithaca, N. Y., January 5, 1898.*

## EDITORIALS.

Dr. Lovett's article on Lie's Transformation Groups came too late for publication in this issue.

Through the courtesy of T. J. McCormack, assistant editor of the Open Court, we were enabled to present in the January number, a portrait of Leonhard Euler. A portait and biography of Euler appeared in the November number of the Open Court.

We note with pleasure that our valued contributor Dr. G. A. Miller has been appointed instructor in mathematics at Cornell University. His work began the first of this year. The *Cornell Era* of Feb. 5th expresses high appreciation of the fact that Cornell has been fortunate enough to secure so valuable an addition to its Faculty of mathematical instructors. Dr. Miller is a young mathematician of great promise. In the summer of 1895 he went to Germany and spent one year almost entirely in working with Professor Lie, the following year he spent at Paris working with Professor Jordan. That Dr. Miller has done some very fine work in the subject of groups, is sufficiently attested by the fact that both Jordan and Picard have presented his communications to the Paris Academy of Science.

## *BOOKS AND PERIODICALS.*

*College Algebra.* By Edward A. Bowser, LL. D., Professor of Mathematics and Engineering in Rutgers College, New Brunswick, N. J. 558 pages. Boston: D. C. Heath & Co.

This work has become so far established in favor that it needs no special commendation from us. In matters, arrangement, and manner of treatment this book has numerous good features. It has just been adopted as the text to be used in the new Cosmopolitan University correspondence course. J. M. C.

(1) *Elements of Calculus.* By James M. Taylor, A. M., Professor of Mathematics, Colgate University. 249 pages. Boston: Ginn & Company.

(2) *Elements of Determinants.* By Paul H. Hanus, Assistant Professor of

History and the Art of Teaching, Harvard University. 217 pages. Boston: Ginn & Company.

(1) Prof. Taylor's Calculus is a deservedly popular text-book. By reason of the many illustrations of the elementary processes of the Calculus, it is admirably adapted to the needs of those commencing the subject. Throughout the book many practical problems are given, which serve to exhibit the power and use of the science, and to arouse and keep alive the interest of the student. *Prof. Taylor's Calculus* has also been selected as the text in the Cosmopolitan University Course.

(2) This well-known work, while suited to the needs of the class-room, is especially adapted to self-instruction. The first presentation of the subject is made with great simplicity, but as the student advances less attention is given to details. While the treatise is not voluminous, yet enough is given to show something of the power and utility of determinants and the consequent importance of the study. J. M. C.

*Exercises in Choice and Chance.* By William Allen Whitworth, M. A., Late Fellow of St. John's College, Cambridge. Price, 6s. 1897. Cambridge: Deighton, Bell & Co.

Prof. Whitworth's book of 700 exercises includes hints for the solution of all the questions in his well-known work on "Choice and Chance," with introductory chapters on the Summation of Certain Series, and a Greshan Lecture on Applications of the Laws of Chance. There are interesting notes on many of the solutions, and the collection of exercises illustrates nearly all the principles and methods arising in questions in probability. Those of our readers who are acquainted with the author's charming little treatise on "Choice and Chance" will note the appearance of the book under review with great satisfaction. J. M. C.

*Through Quadratico Equations.* By Jos. V. Collins, Ph. D., Professor of of Mathematics, in State Normal School, Stevens Point, Wis. 8vo. Cloth. 85+41 pages. Chicago: Scotts, Foresman & Company.

In the publication of this Manual, the author has availed himself of the opportunity to present in an admirable way some suggestions for the study and teaching of Algebra. Dr. Collins Text-book of Algebra has been adopted in the State of Kansas and it was due to this fact that the publication of the Manual was made necessary. The Manual contains many historical notes of great interest in addition to much original matter. Dr. Collins is a strong advocate of the disuse of the cumbersome radical sign. See his article in MONTHLY, Vol. II, No. 4. B. F. F.

*Analytic Functions.* Suitable to Represent Substitutions. By Leonard E. Dickson, Ph. D. Quarto Pamphlet, 10 pages.

The above is a reprint from the American Journal of Mathematics and was written while Dr. Dickson was pursueing his coarse of mathematics in the University of Chicago.

*The Analytic Representation of Substitutions on a Power of Prime Number of Letters with a discussion of the Linear Group.* A dissertation presented to the faculty of Arts, Literature and Science, of the University of Chicago for the degree of Doctor of Philosophy. By Leonard Eugene Dickson.

This dissertation is on a subject in which Dr. Dickson is a recognized authority, both in America and Europe. In this thesis he has generalized the results in his article referred to above, and in this wholy original work Dr. Dickson has earned with great credit the honor that the University has conferred upon him, and that at an exceedingly early age, he being no older than twenty-two at the time he received his degree.

*On Rational Quadratic Transformations.* By H. W. Haskell, Ph. D., Associate Professor of Mathematics, University of California.

This paper is contained in the Proceedings of the California Academy of Sciences, February, 1898. The Quadratic Cremona Transformation. By L. E. Dickson, Ph. D., Instructor in Mathematics, University of California.

*The American Monthly Review of Reviews.* An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single number, 25 cents. The Review of Reviews Co., 13 Astor Place, New York.

The March number of the *American Monthly Review of Reviews* is another achievement in monthly journalism. The topics treated in this magazine are such as occupy much space in the daily press, but the *Review* is able to treat them more deliberately and in a more carefully adjusted proportion. No other illustrated monthly appearing on the first day of March will have so much as a reference to the De Lome letter, the *Maine* disaster, or the Zola trial in Paris; but these great themes of the hour are fully discussed in the *Review's* pages. The *Review's* readers expect to have them discussed there, so accustomed have they become to the essential qualities of timeliness and comprehensiveness in the "busy man's magazine."

*The Open Court.* A Monthly Magazine. Devoted to Science of Religion, the Religion of Science, and the extension of the Religious Parlament Idea. Edited by Dr. Paul Carus; Assistant Editor, T. J. McCormack and Associate Editors E. C. Hegeler and Mary Carus. Price $1.00 per year in advance. Single copies, 10 cents.

The December number (1897) contains biography of Lagrange, by Assistant Editor T. J. McCormack. The Frontispiece of this number contains an excellent portrait of that master mathematician. The Jan. number contains a biography and portrait of Laplace. Mr. McCormack has gone to a great deal of trouble and expense in securing portraits of the great masters in mathematics, and it is probable that he can furnish, at a reasonable price, the portraits of most of these great men to any of our readers who may desire them. Other biographies and portraits will appear in future numbers of the *Open Court*.

SOME ERRATA IN JANUARY NUMBER.

Page 14, line 11, for "$(2m-)$" read $(2m-1)$.
Page 15, line 9, for "6/7" read 7/6.
Page 17, line 13, for "Hewlett" read Woodmere; line 15, for "forms" read form; line 21, for "0.04−" read 0.04+; line 22, for "+.13" read 8.13, and for "error+*" read error*; line 25, for "−2.0316+" read −3.0416+.
Page 22, line 20, supply .04 so as to read −3.04; line 22, for "−3.−04, +3.04.
Page 25, line 14, in numerator, for "$n^2[x(n^2-1)+1p]$" read $n^2[q(n^2-1)+2p]$; line 17, for "Then" read Take.
Page 27, line 21, for "horison" read horizon; in figure, join $BM'$, for "$S$" at right read $S''$ and supply $M$ in $M'MM''$.
Page 28, line 6, for "81° 36° 29″" read 81° 36′ 29″; the figure should be drawn so that $ACB$ pass through $M$, and $E$ should be on $RFT$.
Page 29, problem 68, line 1, $x$ should stand in index position with respect to $a$, and in line 2, the small plus signs should be raised to intermediate position, and where "$k-1-1$" occurs, last −1 should be lowered to line of these signs; line 17, for "State University" read "University of Oregon."
Page 30, line 1 of No. 60, for "three" read six; line 2 of No. 60, for "five times" read five-half times, and for "three" read six.